The Light Quark Masses from Lattice Gauge Theory

Brian J. Gough,[a,*] George M. Hockney,[a] Aida X. El-Khadra,[b] Andreas S. Kronfeld,[a] Paul B. Mackenzie,[a]
Bart P. Mertens,[c] Tetsuya Onogi,[d] and James N. Simone[a]
[a] *Fermilab, PO Box 500, Batavia, Illinois 60510*
[b] *Dept. of Physics, University of Illinois, Urbana, Illinois 61801*
[c] *Dept. of Physics, University of Chicago, Chicago, Illinois 60637*
[d] *Dept. of Physics, University of Hiroshima, Higashi-Hiroshima 739, Japan*
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We investigate the masses of the light quarks with lattice QCD. We show that most of the large dependence on the lattice spacing a observed in previous determinations using Wilson fermions is removed with the use of an $\mathcal{O}(a)$ corrected action. In the quenched approximation, we obtain for the strange quark $\overline{MS}$ mass $\overline{m}_s(2\text{ GeV}) = 95(16)$ MeV, and for the average of the up and down quark masses $\overline{m}_l(2\text{ GeV}) = 3.6(6)$ MeV. Short distance arguments and existing staggered fermion calculations make it likely that the answers including the effects of quark loops lie 20% to 40% below this: $\overline{m}_s(2\text{ GeV})$ in the range 54–92 MeV, and $\overline{m}_l(2\text{ GeV})$ in the range 2.1–3.5 MeV. We argue that almost all lattice determinations of the light quark masses are consistent with these values. These low values are outside the range usually given by conventional phenomenology.

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Among the most important applications of lattice gauge theory to particle physics are the calculations required to determine the fundamental parameters of the quark sector of the standard model. One of the most important of these is the overall scale of the light quark masses. It is one of the least well known of the fundamental parameters of the standard model. (Estimates for the strange quark mass range from 100 to 300 MeV for the $\overline{MS}$ masses renormalized at a "high" energy scale, 1 GeV, and for the average light quark mass from 3.5 to 11.5 MeV [1].) It is also one for which lattice methods are almost uniquely reliable, unlike quark mass ratios or the strong coupling constant α_s, for which other powerful methods exist. Values for quark masses have been obtained since almost the beginning of lattice phenomenology [2,3]. However, improved understanding of perturbation theory and finite lattice spacing errors has been required to make sense of the various lattice determinations, which initially ranged over a factor of three.

Lattice determinations of standard model parameters consist of two pieces. Calculations of experimentally measurable quantities such as hadron masses are used to fix the bare coupling constant and quark masses in the lattice Lagrangian. Short distance calculations are used to relate the bare parameters in the lattice theory to renormalized, running coupling constants and masses, such as those of the $\overline{MS}$ scheme.

Quark masses are most easily obtained in lattice calculations by matching pseudoscalar meson masses with experiment. These are among the easiest lattice calculations, having small statistical and finite volume errors. Experimental uncertainties are also negligible. Uncertainties are dominated by truncation of perturbation theory and discretization errors, and by errors arising from the omission of light quark loops (the "quenched" approximation).

The short distance calculations relating the parameters in various regulators may be performed by demanding that short distance quantities such as the heavy-quark potential or current correlation functions be the same in both regulators. It is desirable to do the lattice part of such calculations nonperturbatively as much as possible, to test for the presence of nonperturbative short distance effects and possible poor convergence of perturbation theory. Such nonperturbative short distance analysis for quark masses is currently less advanced than the analogous investigations for the strong coupling constant.

Perturbative relations between the lattice bare mass, m_0, and the $\overline{MS}$ mass, $\overline{m}$, may be obtained by demanding that on-shell Green's functions calculated with both regulators be equal. Analogous perturbative expressions for the renormalization of α_s were initially rendered almost useless by sick behavior in the lattice perturbation series. In Ref. [4] it was shown that such behavior could be understood and mostly eliminated by a mean field theory resummation of large "tadpole" graphs. The effects of such large tadpoles are much less important for quark mass renormalizations than for α_s [5].

To reduce the effects of such graphs further, the expression giving $\overline{m}$ from m_0 may be rewritten in terms of a mean field improved mass $\tilde{m}$,

$$\overline{m}(\mu) = \tilde{m}\left[1 + \alpha_s\gamma_0\left(\ln\tilde{C}_m - \ln(a\mu)\right)\right], \qquad (1)$$

where $\gamma_0 = 2/\pi$ is the leading quark mass anomalous dimension, and $\ln\tilde{C}_m$ is the result of a one loop calculation. Here we use $\tilde{m} = m_0/\sqrt[4]{\langle U_P\rangle}$ for the mean-field-improved bare mass $\tilde{m}$. The nonperturbative value of the plaquette expectation value $\langle U_P\rangle$ is used in the expression for $\tilde{m}$ to incorporate an estimate for higher order tadpole graphs. The one loop term $\ln\tilde{C}_m$ is then adjusted to remove the one loop part of this expression $\sqrt[4]{\langle U_P\rangle} = (1 - (\pi/3)\alpha_s)$.

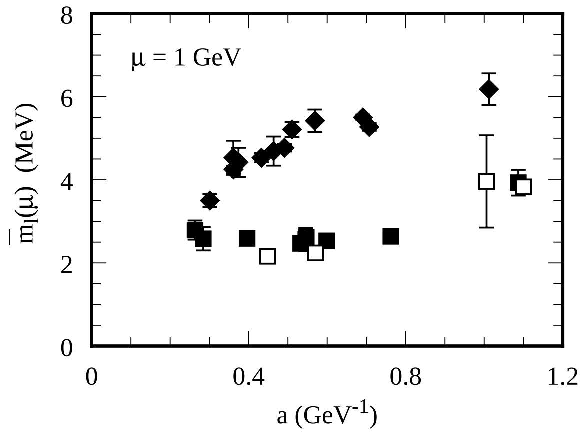

FIG. 1. Previous lattice results for the $\overline{MS}$ masses of the light quarks, renormalized at 1 GeV, with the lattice spacing set by the rho mass. Lattice spacing dependence is large for quenched Wilson fermions (diamonds) and small for quenched staggered fermions (filled squares). Results from two-flavor staggered fermion QCD (open squares) lie below those from quenched approximation staggered fermions by a reasonable amount. Data from Ukawa [2].

In Fig. 1 we show a compilation of previous results given by Ukawa [2]. Quenched results obtained with staggered fermions are almost cut-off independent for lattice spacings less than 1 GeV^{-1}. However, for staggered fermions the constant in Eq. (1) is $\tilde{C}_m = 132.9$ [6]. This leads to correction factors of 50–100%, most of which is unexplained by mean field theory, casting doubt on the reliability of the perturbative relation between the staggered-fermion quark mass and the $\overline{MS}$ quark mass.

For Wilson fermions, we have $\tilde{C}_m = 1.67$ [6], and thus a well-behaved perturbation series. However, the numerical results for the Wilson action show large cut-off dependence. They lie far above the results for staggered fermions but show a downward trend as the lattice spacing is reduced. The Wilson fermion action contains an error of $\mathcal{O}(a)$, which is absent in the staggered fermion action. If the results are extrapolated in a, one obtains a result much closer to the results of staggered fermions. (See, for example, Ref. [8].) However, remaining sources of cut-off dependence are an unknown combination of $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s a)$, $\mathcal{O}(a^2)$, etc. They cannot be estimated or removed by simple extrapolation, since we do not have a quantitative theory of their functional form. One therefore needs to investigate the removal of the dominant $\mathcal{O}(a)$ error from the Wilson action.

A convenient action for doing this has been proposed by Sheikholeslami and Wohlert [9]. Their improved action incorporates an extra dimension five term $\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$, the so-called "clover" term. The one-loop correction to the coefficient of the clover term is large [10], as suggested by mean field theory [4]. It is a three-tadpole correction and can be approximated by $c \approx \langle U_P \rangle^{-3/4}$, where the tree level coefficient is normalized to be one. For the improved action, $\tilde{C}_m = 4.72$ [11]. Thus, Eq. (1) is still well–behaved.

We use this action to determine the overall scale of the light quark masses. (Or equivalently, the coefficient of m_l in the expression $M_\pi^2 = Cm_l + \ldots$. We do not see deviations from the leading order of this equation, see below.) Our lattice spacings range from (the coarse) 1.26 GeV^{-1} (at which uncertainties due to perturbation theory are starting to approach 50%), down to 0.39 GeV^{-1} (where perturbation theory appears well behaved). We have performed the calculation at the largest lattice spacing to investigate its behavior where it is beginning to break down, but we omit it from our final results. The lattice spacings have been obtained from the 1P-1S splitting of the charmonium system, $M_{h_c} - (3M_{J/\psi} + M_{\eta_c})/4$, for which the uncertainties of lattice calculations are particularly small and easy to understand. This means that numerical uncertainties in our results for the quark masses arise from a combination of uncertainties in the charmonium and pion calculations.

We use improved lattice perturbation theory to convert to the $\overline{MS}$ mass at renormalization scale $\mu = 2$ GeV and charmonium splittings to determine the lattice spacing, whereas previous determinations typically used bare perturbation theory at scale $\mu = 1$ GeV and the rho meson mass to determine the lattice spacing. Although renormalization at 1 GeV is conventional in nonlattice results, renormalizing down to such a low scale introduces additional perturbative uncertainty into the results which is not present in the underlying lattice results.

Discussions of our charmonium calculations have appeared in Ref. [12]. Some technical details and results of our pion calculations are given in Table I. For our most significant data point, the improved clover action at $\beta = 6.1$, we have used 100 configurations separated by 4000 heat bath gauge sweeps. Pion correlation functions were calculated using 2×2 correlated fits (fitting two states using two operators for the pions). Contributions from excited states were checked further on the smaller lattices by comparing with 1×1 and 3×3 fits. Statistical errors were calculated using 1000 bootstrap samples. Longer descriptions of our analyses for the charmonium system and for the light quark masses are in preparation [13].

In Fig. 2 and in Table I we show our results for the light quark masses in the quenched approximation. The errors shown are statistical only. The diamonds are our results for unimproved Wilson fermions. They are consistent with the existing determinations (diamonds in Fig. 1). The triangles are our results for the mean-field-improved clover action. Most of the cut-off dependence has been removed.

Remaining sources of such cut-off dependence could include large α_s^2 corrections to the mass relation, Eq. (1), further corrections to the clover coefficient in the pion numerical calculations, and $\mathcal{O}(a^2)$ corrections to the charmonium 1P-1S splitting. $\mathcal{O}(a)$ corrections are expected to be negligible for this splitting, but $\mathcal{O}(a^2p^2)$ corrections could be larger since quark momenta are larger in charmonium than in pions. We estimate $\mathcal{O}(a^2p^2)$ corrections in the charmonium splitting to range from 4% to 20% on our three finest lattice spacings. The perturbative one-loop result for the coefficient of the $\mathcal{O}(a)$ clover correction agrees with the mean field estimate [10]. However, a nonperturbative determination appears indeed to favor a further significant correction [14]. Purely perturbative errors in the relation between the lattice and $\overline{MS}$ masses should be of order $\alpha_s^2 \sim 5\%$ at our finest lattice spacing. Other smaller uncertainties include finite volume effects, which are expected to be a couple of per cent or less, and statistical errors, which are 4% and arise mostly from the lattice spacing derived from the charmonium system.

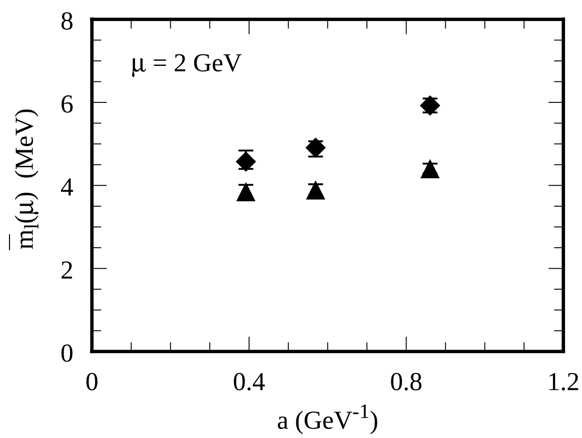

FIG. 2. Our results for the masses of the light quarks. Most of the lattice spacing dependence of unimproved Wilson fermions (diamonds) is removed by the use of an $\mathcal{O}(a)$ corrected action (triangles) with a tadpole improved coefficient. The lattice spacing is set by the charmonium 1P-1S splitting. Errors are statistical only.

We have examined the pseudoscalar meson mass squared as a function of the quark mass, which should be linear plus small corrections in the small quark mass limit. Our numerical data are for quark masses in the range $0.4m_s$ to m_s. In this range, we find no statistically significant evidence for quadratic terms in M_π^2 vs. m_l, much less the very large quadratic terms that have been postulated to make $m_u = 0$ consistent with experiment. Therefore, our results for the ratio of the strange to light quark masses agree with lowest order chiral perturbation theory: $(m_s + m_l)/(2m_l) \approx M_{K^0}^2/M_{\pi^0}^2 \approx 13.6$. The values for m_l are obtained by linear extrapolation from the lowest masses at which we have performed simulations down to the physical mass value.

At present, the uncertainty associated with the remaining cut-off dependence is the least reliably understood uncertainty in the quenched approximation. Pending further understanding of this error, we take our result at the smallest lattice spacing as the top of our lattice spacing error bar. We take a linearly extrapolated result (the lower of two plausible extrapolation methods) through the three finest lattice spacings as the bottom. This gives a range of 0.8 MeV for the cut-off dependence uncertainty, and we take the center of this range as our continuum limit, quenched approximation result:

$$\overline{m}_l(2\text{ GeV}) = 3.6(6)\text{ MeV}, \qquad (2)$$

$$\overline{m}_s(2\text{ GeV}) = 95(16)\text{ MeV}. \qquad (3)$$

The perturbative and cut-off dependence uncertainties were added linearly in the total error, since they are related. All other uncertainties were added in quadrature.

Another determination of the strange quark mass with an $\mathcal{O}(a)$ improved action has been reported [15]. This determination used a tree-level, rather than a mean-field improved, estimate for the clover coefficient. They obtained $\overline{m}_s(2\text{ GeV}) = 128(18)$ MeV. They did not attempt to correct for the effects of the remaining lattice spacing dependence or the effects of the quenched approximation. Most of the discrepancy with our results arises from fact that we have used much larger clover coefficients, and make an allowance for the fact that we continue to find significant cut-off dependence even so.

In the quenched approximation, QCD couplings run slightly incorrectly. The strong coupling constant, for example, runs too fast without the effects of light quark loops [12]. To leading logarithmic accuracy, $\alpha_s(\pi/a)$ is too small by a factor of about $\beta_0^{(3)}/\beta_0^{(0)}$, where $\beta_0^{(0)}$ and $\beta_0^{(3)}$ are the leading quenched and unquenched β functions, respectively. This means that the running of the quark mass in the perturbative momentum region around π/a is too slow, by about the same factor. In Ref. [16], the ratio of quenched and unquenched quark masses arising from the perturbative region was estimated, to leading logarithmic accuracy, to be

$$\frac{m(\pi/a)|_{\text{qu.}}}{m(\pi/a)|_{\text{unqu.}}} \approx \alpha_s(\pi/a)^{\frac{\gamma_0}{2}(1/\beta_0^{(0)} - 1/\beta_0^{(3)})} \qquad (4)$$

$$\approx 1.15 \text{ to } 1.20, \qquad (5)$$

for $\alpha_s(\pi/a) \approx 1/6$ to $1/8$. There is, of course, an additional contribution from the nonperturbative region, which is unknown. However, a correction due to light quark loops of tens of per cent in the downward direction from the perturbative region at least would not be unexpected.

Some quenched and unquenched staggered results summarized in Ref. [2] are shown in Fig. 1. (Unquenched Wilson fermion calculations appear to be much more difficult to perform and harder to interpret.) The unquenched results indeed lie below the quenched results by roughly the expected amount, and we take them seriously enough to use them to estimate the effects of quenching. We argued above that quenched staggered quark mass determinations look good in most ways, but are unreliable because of the poor convergence of perturbation theory. However, the large corrections cancel out in the ratio of the quenched and unquenched determinations, making this a useful quantity to examine. To minimize effects due to differences in analysis methods, we estimate the ratio from the results of a single group, at similar volumes and lattice spacings (about 0.4 GeV^{-1}) [17,18], and obtain

$$\frac{\overline{m}_l(1.0\ \mathrm{GeV})_{\mathrm{n_f}=0}}{\overline{m}_l(1.0\ \mathrm{GeV})_{\mathrm{n_f}=2}} \approx \frac{2.61(9)}{2.16(10)} \tag{6}$$

$$= 1.21(7). \tag{7}$$

Since there are, in fact, three flavors of light quarks in the world and not two, we will use this ratio as a lower bound on the actual ratio and use the square (corresponding to four light quarks) as an upper bound.

In summary, after making some plausible cuts, existing determinations are reasonably consistent, or have plausible explanations for discrepancies. We omit results with very small physical volumes (smaller than 1.5 fm) and very large lattice spacings (larger than 0.2 fm, or 1.0 GeV^{-1}). We also do not attempt to interpret the results with unquenched Wilson fermions, which are in a more primitive state than those with staggered fermions. Of the remaining determinations, we have shown that the cut-off dependence and large size of determinations with quenched Wilson fermions arise mostly from the well-known $\mathcal{O}(a)$ error. The remaining discrepancy between the quenched clover-improved fermion results and the quenched staggered fermion results is plausibly attributed to the apparent poor convergence of staggered fermion perturbation theory and the remaining cut-off dependence in the improved fermion results. The small difference between quenched and unquenched staggered fermion results is roughly what is expected. Putting all this together, we arrive at the following estimates for the light quark masses including effects of light quark loops, which we believe are consistent with all known facts:

- $\overline{m}_s(2\ \mathrm{GeV})$ in the range 54–92 MeV,
- $\overline{m}_l(2\ \mathrm{GeV})$ in the range 2.1–3.5 MeV,

for the $\overline{MS}$ masses renormalized at 2 GeV. These estimates arise from combining our quenched result, Eq. (2), with the correction ratio obtained from staggered fermions, Eq. (7). Renormalizing down to the scale 1 GeV, where conventional mass estimates are often quoted, the estimates are raised by 10%, to $\overline{m}_s(1\ \mathrm{GeV})$ in the range 59–101 MeV, and $\overline{m}_l(1\ \mathrm{GeV})$ in the range 2.3–3.9 MeV.

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* Present address: Los Alamos National Laboratory, Los Alamos, New Mexico

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TABLE I. Our results for $\overline{m}_l(2\ \mathrm{GeV})$, the average of the u and d quark masses, renormalized at 2 GeV. c is the coefficient of the $\mathcal{O}(a)$ improvement operator.

β	5.5	5.7	5.9	6.1
a (GeV^{-1})	1.26	0.86	0.57	0.39
volume	$8^3 \times 16$	$12^3 \times 24$	$16^3 \times 32$	$24^3 \times 48$
m ($c = 0$)	6.31(26)	5.93(17)	4.88(18)	4.62(22)
m (improved)	4.75 (19)	4.41(12)	3.90(13)	3.84(18)
c	1.69	1.57	1.50	1.40