

# Effects of Improvement: $f_\pi$ and $m_q$

Brian Gough, Tetsuya Onogi, Jim Simone

Theoretical Physics Group, Fermilab, PO Box 500, Batavia, IL 60510, USA

We present a study of the effects of improvement on light-quark physics using the Fermilab formalism. The calculations were performed at three different lattice spacings, with the SW action and traditional Wilson action (both tadpole improved). We find that  $O(a)$  effects for the decay constant  $f_\pi$  and quark mass  $m_q$  can be successfully removed using the tadpole-improved SW action in the light-quark regime.

## 1. Introduction

In this project, we have made a systematic comparison of improved and unimproved actions by studying the lattice-spacing dependence of the simplest quantities in light-quark physics: the pion decay constant,  $f_\pi$ , and the (isospin-averaged) light-quark mass,  $m_q$ .

The first step in improving the Wilson action for QCD is to remove the leading  $O(a)$  errors from Wilson fermions. Following Symanzik, this leads to the Sheikholeslami-Wohlert (SW)  $O(a)$ -improved fermion action [1]. With tadpole improvement [2], this gives a QCD action which is essentially correct to  $O(a^2)$ .

A reliable improved action for light quarks would allow unquenched calculations on small lattices, with important implications for phenomenology.

## 2. Lattice Details

We study scaling effects using three lattices, at  $\beta = 5.7, 5.9$ , and  $6.1$ , which cover a range of more than a factor of two in  $a$ .

On each lattice we compute Wilson and SW propagators, as described below. In this study we aim to test the effects of Symanzik improvement, i.e. the presence or absence of the SW clover  $O(a)$ -improvement term in the action. We do not investigate the effects of tadpole improvement compared with no tadpole improvement — tadpole improvement is used throughout, and the same tadpole-improvement procedure is used for both actions.

The full lattice details are shown in Table 1.

Table 1  
Lattice details ( $n_f = 0$ )

|                     | $\beta=6.1$      | $\beta=5.9$      | $\beta=5.7$      |
|---------------------|------------------|------------------|------------------|
| $L^3 \times T$      | $24^3 \times 48$ | $16^3 \times 32$ | $12^3 \times 24$ |
| $a^{-1}$            | 2.43 GeV         | 1.78 GeV         | 1.15 GeV         |
| $L^{phys}$          | 2.0 fm           | 1.8 fm           | 2.1 fm           |
| $c$                 | 1.46*            | 1.50*            | 1.57             |
| # configs           |                  |                  |                  |
| Wilson              | 100              | 100              | 200              |
| SW                  | 100              | 150              | 200              |
| # light $\kappa$ 's | 5                | 5                | 5                |

\*For historical reasons, the  $\beta = 6.1$  and  $5.9$  results were obtained with the mean-field value  $c = 1.4$ . The differences are negligible and do not modify the conclusions.

## 3. Improvement Procedure

We use a tadpole-improved SW fermion action,

$$S = S_{\text{Wilson}} + (i/2) c \bar{\psi} \sigma \cdot F \psi \quad (1)$$

where the coefficient of the clover term,  $c$ , is set non-perturbatively,  $c = 1/(u_0)^3$ . The average link,  $u_0$ , is determined from the plaquette,

$$u_0 = \langle (1/3) \text{Tr} U_p \rangle^{1/4} \quad (2)$$

to give the values of  $c$  shown in Table 1. The original tree-level SW action used  $c = 1$ .

For the fermions, we take the tadpole-improved field  $\tilde{\psi}$ ,

$$\tilde{\psi} = \sqrt{1 - 3\kappa/4\kappa_{crit}} \psi \quad (3)$$

which has the correct normalization in the chiral limit,  $\tilde{\psi} = \psi/2$ .

In calculating matrix elements with the SW action an improved operator must be used [3]. In the formalism of Heatlie *et al* this is equivalent to rotating the quark fields,

$$\psi \rightarrow [1 - (z \not{D} - (1 - z)m_0)/2] \psi. \quad (4)$$

Choosing  $z = 0$ , the rotation becomes,

$$\psi \rightarrow [1 + m_0/2] \psi \quad (5)$$

and its effects will vanish in the chiral limit, where  $m_0 = 0$ .

With the Fermilab formalism the corresponding rotation is,

$$\psi \rightarrow \sqrt{1 + m_0} [1 + d_1 \vec{D} \cdot \vec{\gamma}] \psi \quad (6)$$

but its effects also vanish in the chiral limit (since  $d_1 \sim m_0$  for small  $m_0$ ). For convenience we used this trick to obtain our decay-constant results from the ordinary unrotated fields.

In chiral extrapolations we use linear fits in the lattice pole mass,

$$m_q = \ln(1 + (1/2\kappa - 1/2\kappa_{crit})) \quad (7)$$

to model the  $\kappa$ -dependence. This definition automatically resums the tree-level large- $m$  corrections in the quark mass.

For the Wilson action we use exactly the same tadpole-improvement procedure for the fermions, but set  $c = 0$ .

#### 4. Setting the Scale

In a systematic study of  $O(a)$  effects it is important to choose a scale which is as free from  $a$ -dependence as possible, so that other quantities are not affected by a common variation in the scale.

We set the scale using the spin-averaged  $1P - 1S$  splitting from charmonium [4]. The spin-averaged splitting is found to be insensitive to  $O(a)$  corrections from the clover term, and the resulting scale is free from  $O(a)$  effects. Using this scale, the  $a$ -dependence of other quantities can be determined directly, up to the remaining  $O(a^2)$  systematic errors in the scale.

The advantage of using charmonium to set the scale is that it is possible to separate the  $a$ -dependence of the quantities under study from the  $a$ -dependence of the scale (known to be good to  $O(a^2)$ ). Other possible scales, such as  $m_\rho$ , have the disadvantage that the scale itself can contain an  $O(a)$  effect.

#### 5. Multistate Fitting

When studying small  $O(a)$  effects it is important to keep systematic errors under control. These errors may have an  $a$ -dependence which conceals the true scaling behavior of the quantities under study. In particular, excited-state contamination can introduce large  $O(a)$  effects, purely from the systematic variations in a poor fitting procedure.

We use a sophisticated multistate fitting program to eliminate excited-state contamination. A set of optimised smearing functions is available on each lattice, created from a study of coulomb-gauge pion wavefunctions, typically for  $1S$  and  $2S$  sources. The fitting is performed using a matrix of correlators,  $\delta$ - $\delta$  (point source),  $1S$ - $1S$ ,  $2S$ - $2S$ ,  $A_4$ - $\delta$ ,  $\dots$ , and a choice of one-exponential or two-exponential fits. We check for consistency between the different fitting methods, to ensure that excited-state contamination has been removed.

#### 6. Decay Constant

Previous studies of  $f_\pi$  using Wilson fermions have all shown a negligible dependence on lattice-spacing. In order to determine whether this is due to an intrinsically small  $O(a)$  correction for  $f_\pi$  from the action, or a cancellation by  $O(a)$  effects from excited-state contamination, our analysis was carried out using multistate fits.

Fermilab multistate results for  $f_\pi$  using the tadpole-improved Wilson and tadpole-improved SW actions are shown in Fig.(1), with the physical value  $f_\pi = 132$  MeV. The lattice values are also extrapolated to the continuum limit using a simple linear fit in  $a$ .

The Wilson results are renormalized using the tadpole-improved  $Z_A$  of Lepage and Mackenzie [2]. The tadpole-improved value of  $Z_A$  for the

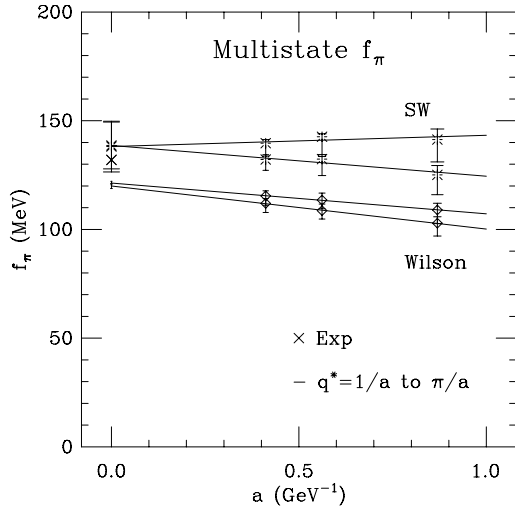


Figure 1. Comparison of SW and Wilson  $f_\pi$ . The lines show the effect of varying  $q^*$  from  $1/a$  (lower line) to  $\pi/a$  (upper line).

local current with the SW action was obtained from the work of Borrelli *et al* [5], by subtracting the tadpole contribution from their intermediate results,

$$Z_A(q^*) = 1 - 0.31\alpha_V(q^*) \text{ (Wilson)} \quad (8)$$

$$= 1 - 0.61\alpha_V(q^*) \text{ (SW)}. \quad (9)$$

The exact values of  $q^*$  remain to be determined by analytic calculations. For the moment, we allow  $q^*$  to vary over a reasonable range, from  $1/a$  to  $\pi/a$ , and regard this as a systematic error.

The additional systematic errors from the  $1P-1S$  charmonium scale are expected to be  $O(p^2a^2)$ , which at  $\beta = 5.7$  might be 5-10%.

Allowing for these systematic errors, the multistate fits confirm the behavior of the existing Wilson data, and show a small lattice-spacing dependence. This indicates that the coefficient of the  $O(a)$  correction to Wilson  $f_\pi$  is intrinsically small.

The SW results are consistent with a small or no  $O(a)$  effect within systematic errors. The addition of the  $O(a)$  correction term to the Wil-

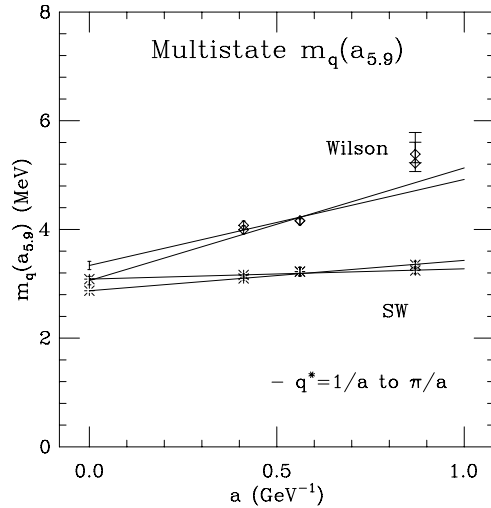


Figure 2. Comparison of SW and Wilson  $m_q$ . The lines show the effect of varying  $q^*$  from  $1/a$  (lower line at large  $a$ ) to  $\pi/a$  (upper line at large  $a$ ).

son action does not increase  $a$ -dependence of the results when the  $a$ -dependence is already small. This is a requirement of improvement.

Even allowing for the uncertainty in  $q^*$ , the linear extrapolations of Wilson and SW  $f_\pi$  to the continuum limit do not meet. This is a possible indication of the perturbative ( $g^4$ ) or quadratic ( $a^2$ ) corrections to the naive linear scaling law.

## 7. Quark Mass

We extract the light-quark mass from the pion mass chiral extrapolation (by determining the quark mass that gives the physical pion mass),

$$m_\pi^2 = Am_q. \quad (10)$$

In Gupta's Lattice '94 review of quark masses a compilation of existing Wilson data showed a large variation of  $m_q$  with lattice-spacing, and a significant discrepancy between Wilson and Staggered results [6].

Again, in order to determine whether this is due to systematic  $O(a)$  effects from excited-state

contamination, our analysis was carried out using multistate fits.

The Fermilab multistate results for  $m_q$  using tadpole-improved Wilson and tadpole-improved SW actions are shown in Fig.(2). The lattice  $m_q$  values are converted to a common scale of  $\beta = 5.9$  using the 1-loop relation,

$$m(a_{5.9}) = (1 - (2/\pi)\alpha(q^*) \ln(a/a_{5.9})) m(a). \quad (11)$$

This removes the lattice-spacing dependence due to the running of the mass. We keep the masses in the lattice scheme. An additional constant factor,  $Z_{\overline{MS}}$ , would be needed to obtain the conventional  $\overline{MS}$  value. Since we are only interested in the scaling properties this overall constant is not important here.

The masses are again extrapolated to the continuum limit using a simple linear fit in  $a$ . In this case the Wilson and SW extrapolations are in good agreement.

Multistate fitting confirms the presence of a large  $O(a)$  effect in Wilson  $m_q$ . The SW results show a small  $O(a)$  effect, compatible with zero within systematic errors. This indicates that the discrepancy between the existing Wilson and Staggered results is probably due to a large  $O(a)$  error from the Wilson action. The Staggered action, which is also correct to  $O(a^2)$ , should be in agreement with the tadpole-improved SW action.

## 8. Conclusions

For the decay constant, multistate fitting confirms the small  $O(a)$  effects found in Wilson  $f_\pi$ . The  $O(a)$  effects for the improved SW  $f_\pi$  are also small, within systematic errors from  $q^*$  and  $p^2 a^2$  corrections to the charmonium scale. There is a difference between the continuum extrapolations of SW and Wilson  $f_\pi$  which suggests perturbative or quadratic corrections to naive linear scaling.

For the quark mass, multistate fitting confirms the large  $O(a)$  effects found in Wilson  $m_q$ . The  $O(a)$  effects for the SW  $m_q$  are consistent with zero, within systematic errors. The improved action removes the  $a$ -dependence of  $m_q$  even down to  $\beta = 5.7$ , when using the clover term with tadpole improvement.

Further simulations at lower  $\beta$  values are desir-

able, to determine the scale at which  $O(a^2)$  errors become important. At this stage there is no clear evidence of  $O(a^2)$  effects, even at  $\beta = 5.7$ . However, for larger lattice spacings it is essential to first obtain the correct  $q^*$  values, since the systematic error due to  $q^*$  increases as the perturbative corrections increase.

The costs involved in using the SW action are small compared with the benefits of reducing  $O(a)$  errors. On the CANOPY/ACPMAPS platform the addition of the clover term typically increases the runtime of the quark inversion by approximately 30%.

In summary, when Wilson results already show small  $O(a)$  effects, improvement does not make things worse, and when Wilson results show large  $O(a)$  effects, improvement reduces them. Thus, use of the tadpole-improved SW action is recommended.

## 9. Acknowledgments

We wish to thank our collaborators, and our colleagues in the Fermilab Computing Division. These calculations were performed on the Fermilab ACPMAPS supercomputer. Fermilab is operated by Universities Research Association, Inc., under contract DE-AC02-76CH03000 with the U.S. Department of Energy.

## REFERENCES

1. B. Sheikholeslami and R. Wohlert, Nucl. Phys. **B259**, 572 (1985).
2. G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **48**, 173 (1991).
3. G. Heatlie, G. Martinelli, C. Pittori, G. C. Rossi, and C. T. Sachrajda, Nucl. Phys. **B352**, 266 (1991).
4. A. X. El-Khadra, G. Hockney, A. S. Kronfeld, and P. B. Mackenzie, Phys. Rev. Lett. **69**, 729 (1992).
5. A. Borrelli, C. Pittori, R. Frezzotti, and E. Gabrielli, Nucl. Phys. **B409**, 382 (1993).
6. R. Gupta, Nucl. Phys. B (Proc. Suppl.) **42**, 85 (1995).